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Technical Notes

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Aerodynamic Model Reduction Through Balanced Realization

John A. Rule*
Duke University, Durham, North Carolina 27708
David E. Cox†
NASA Langley Research Center, Hampton, Virginia 23681
and
Robert L. Clark‡
Duke University, Durham, North Carolina 27708

Introduction

AERODYNAMIC model reduction based on internal balancing is examined for incompressible potential flow and is shown to allow signification reduction in model complexity. The technique was validated against classical two-dimensional incompressible unsteady results, as well as an aeroelastic flutter boundary calculation. In particular, new insight is given into the relationship between balanced aerodynamic states and the fundamental physics of airfoil and wing motion. The form of the principal retained states suggests that the physics of a few dominant limiting-case airfoil motions characterize the flow. Comparison with exact solutions for a two-dimensional case are shown to illustrate this point.

A popular technique for aerodynamic model reduction is to perform a Karhunen–Loève decomposition of the aerodynamic field. An excellent description of this technique can be found in Ref. 1 under the name proper orthogonal decomposition (POD), and some applications of this technique are described in Refs. 2 and 3. Karhunen–Loève decomposition involves the determination of the empirical eigenmodes of a dynamic system. For aerodynamic applications, this is accomplished by taking snapshots of an unsteady flow at some finite number of times, forming an autocorrelation matrix of these snapshots, and calculating the eigenvalues and eigenvectors of the autocorrelation matrix. From this, a set of fluid modes emerges that forms a basis for model reduction. These modes are optimal in terms of energy in the field; however, they do not consider the physical effect of the field, such as the integrated loads driving a flexible airfoil.

Drawn from structural mechanics, some recent model reduction efforts in the field of unsteady aerodynamics have involved formulating the governing equations as a generalized eigenvalue problem.[4–7] Standard eigensystem solution techniques produce a set of eigenvectors, which represent fluid modes, and eigenvalues, which suggest the time behavior of each mode. In aeroelastic analysis, lightly damped modes are considered to be more important to the model, whereas more heavily damped modes are neglected. Unfortunately, practical experience has shown that with this reduction an excessive number of modes must be retained and static corrections added to achieve acceptable results.

The present work follows an approach introduced by Baker[8] and Baker et al.[9] and bases the model reduction on observability and controllability concepts from control theory.[10,11] The key feature of this approach is that the reduced-order model is expressed in terms of coordinates that are optimal to a particular input/output mapping, rather than an energy optimal basis as in the POD, or a basis corresponding to lightly damped behavior as in the eigenanalysis. Aerodynamic systems provide an ideal opportunity for the application of balanced realization techniques because it is frequently the case that a large number of states must be used to transmit information from a small number of inputs (system geometry, control surface position), to a small number of outputs (net lift, moment about elastic axis, drag). This is especially the case with computational fluid dynamics schemes, which require the computation of two or three components of velocity plus pressure and density at thousands of nodes around a body, only to obtain a few integrated forces. From this viewpoint, the details of the flow are unimportant, and models may be considered accurate if they capture the transmission path from geometry to forces.

The concepts of state controllability, state observability, and balanced realization are discussed in the present work with respect to inviscid, incompressible, unsteady aerodynamic models. Emphasis is placed on a new physical interpretation of the resulting balanced states and their correlation with exact flow solutions. Sample calculations for an unsteady aerodynamics problem is presented, along with representative aeroelastic flutter calculations.

Model Development

The aerodynamic method used here is the unsteady, incompressible vortex lattice method developed in Ref. 5. The body is assumed to be a two-dimensional flat plate airfoil, which behaves structurally as a typical section with two degrees of freedom first described by Theodorsen.[12]

The flow is modeled by N equally spaced vortices with M vortices along the airfoil and the remainder in the wake. Strengths of these vortices, γ_i, at time $n+1$ are related to the vortex strengths at the previous time step and the normal wash on the airfoil,

$$\boldsymbol{\Gamma}^{n+1} = \mathcal{A}\boldsymbol{\Gamma}^{n} + \mathcal{B}\boldsymbol{w}^{n+1} \tag{1}$$

where $\mathcal{A} \in \boldsymbol{R}^{N\times N}$ and $\mathcal{B} \in \boldsymbol{R}^{N\times M}$ are constants and $\boldsymbol{\Gamma} = [\gamma_1, \gamma_2, \ldots, \gamma_n]^T$ is a state vector. The input to this system is the normal wash and is fully defined by the airfoil motion in pitch and plunge. Finally, outputs from the aerodynamic model relevant to a coupled aeroelastic system are the moment about the elastic axis, M_α, and the total lift L. These can be calculated as integrals on pressure differential given by Bernoulli's equation:

$$L = \int_{-b}^{b} \rho\left[U\gamma(x) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-b}^{x}\gamma(\hat{x})\,\mathrm{d}\hat{x}\right]\mathrm{d}x$$

$$M_\alpha = \int_{-b}^{b} \rho(x-e)\left[U\gamma(x) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-b}^{x}\gamma(\hat{x})\,\mathrm{d}\hat{x}\right]\mathrm{d}x \tag{2}$$

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*Assistant Research Professor, Department of Mechanical Engineering and Materials Science; currently Lead Engineer, Alphatech, Inc., 6 New England Executive Park, Burlington, MA 01803.
†Senior Engineer, Guidance and Control Branch.
‡Associate Professor, Department of Mechanical Engineering and Materials Science. Associate Fellow AIAA.

The discretization of these integrals yields the expression of lift and moments as linear sums of γ_i and so forms an output equation for the state model:

$$y = [M_\alpha \quad L]^T \tag{3}$$

$$y = \mathcal{C}\Gamma^n + \mathcal{D}w^{n+1} \tag{4}$$

The system described by Eqs. (1) and (4) can be converted to a continuous time state-space model (A, B, C, D) and coupled directly to a structural model for aeroelastic analysis. However, a significant level of model reduction is possible through a transformation to balanced coordinates. In balanced coordinates, a system's observability and controllability gramians are diagonal and equal, that is,[11]

$$G = \int_0^\infty e^{A\tau} B B^T e^{\tau A^T} \,\mathrm{d}\tau \tag{5}$$

$$G = \int_0^\infty e^{\tau A^T} C^T C e^{A\tau} \,\mathrm{d}\tau \tag{6}$$

The transformed states of a balanced realization are orthogonal and ordered according to their contribution to the input/output mapping. In the present context, the first few states provide the strongest transmission path through the aerodynamics from the structural inputs of airfoil position and velocity to the integrated lift and moment acting on the airfoil. A detailed description of an algorithm to find the balancing transformation may be found in Ref. 13 and is implemented in commercially available numerical analysis packages. Note that the computational cost of finding this transformation is $\mathcal{O}(N^3)$, which is comparable in cost to finding the eigenvalues and eigenvectors of the same system. However, this still becomes impractical for large systems, and iterative techniques must be employed to find the balancing transformation.[9,14] The remainder of this Note will examine the accuracy loss incurred by balanced model reduction and a physical interpretation of the vorticity distributions corresponding to the most significant modes.

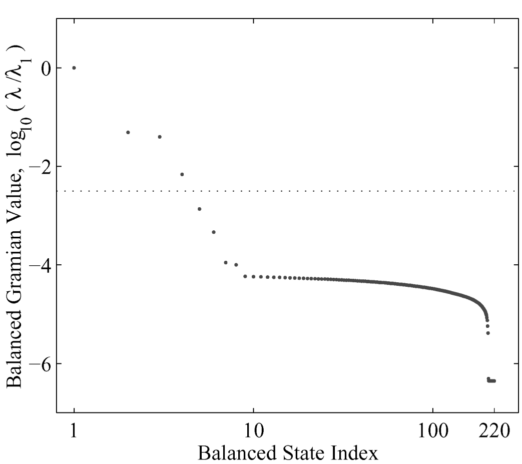

Fig. 1 Magnitude of elements in the balanced gramian matrix, indicating relative importance of each input mode.

Validation of Reduced-Order Model

This section will evaluate the performance of these reduced-order models relative to the full aerodynamic models and also relative to the eigensystem approach of aerodynamic model reduction. Results for the classic Wagner problem are given, as well an aeroelastic flutter calculation. The representative calculations provided here follow those provided in Ref. 5 to facilitate the comparison of the different model reduction methods.

In the aerodynamic model, the airfoil section was modeled with 20 discrete vortices and the wake was modeled with an additional 200 vortices. This model was reduced using balanced realization theory. The values along the diagonal of the balanced gramian matrix are shown in Fig. 1. These values indicate the relative contribution of each state to the output. Model reduction was achieved by applying a truncation criterion, which ensured no more than 1% error in the calculated values of lift and moment. This criterion was met by retaining just 4 of the 220 states in the balanced realization. The dashed line in Fig. 1 illustrates the relative importance of the retained states (above the line) and discarded states (below the line).

Unsteady Aerodynamics

Consider first the Wagner problem, which models the lift response of a two-dimensional flat plate airfoil due to a step change in angle of attack. Figure 2 compares the computed dimensionless lift $\phi(s)$, using the balanced realization technique and using a standard eigensystem approach. In the case of balanced realization, three calculations were performed using the first 5, 25, and 125 principal circulation states, respectively. Resulting lift calculations are equally good in all three cases, suggesting that just the first few states are necessary to capture both transient and steady-state system behavior. The eigensystem solution utilizes the static correction discussed in Ref. 5, with three calculations performed using the first 5, 25, and 125 eigenmodes. With both methods, calculated lift is quite accurate for large time ($s > 4$), but the eigensystem approach is inaccurate for short times. This error depends on the number of retained states in the calculation, and is actually quite small using five states, getting progressively worse as states are added. However, these high-frequency errors disappear if all of the eigenmodes are retained.

This state-dependent error demonstrates the primary drawback of the eigensystem approach. Eigenvalues and eigenvectors are related to the internal dynamics of the mathematical model and not to the physical input/output mapping between airfoil geometric states and integrated forces. Thus, performance of the reduced-order model is not uniformly related the number of states retained.

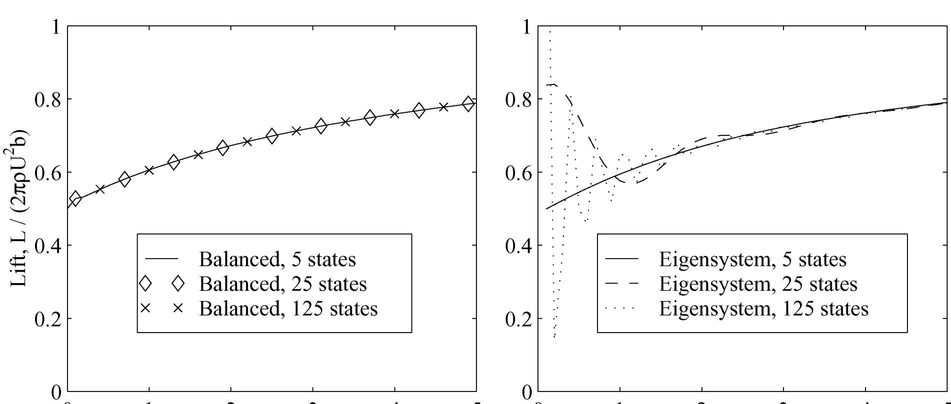

Fig. 2 Comparison of two different model reduction techniques for computing the lift response to a step change in angle of attack (Wagner function).

Table 1 Physical parameters for flutter calculation

Property	Symbol	Value
Mass ratio	μ	20
Frequency ratio	ω_h/ω_α	0.3
Static imbalance	x_α/b	0.2
Radius of gyration	r_α/b	0.5
Elastic axis	e/b	−0.1

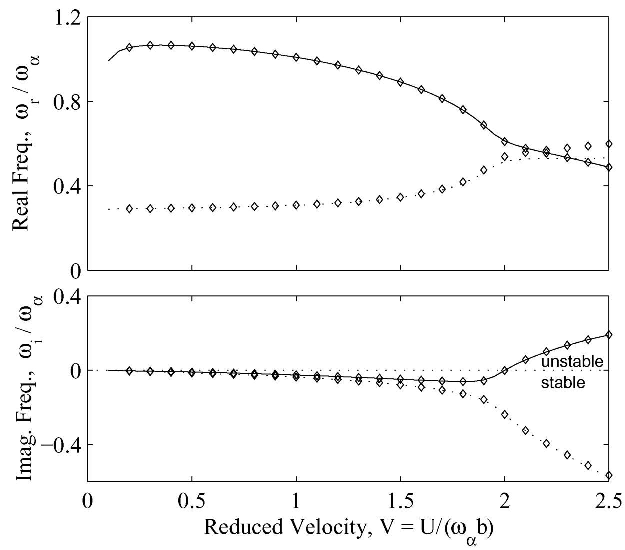

Fig. 3 Natural frequencies and damping of the aeroelastic modes of a typical section model.

Aeroelastic Coupling

The reduced-order aerodynamic model was coupled with a structural model for the flat plate airfoil. This coupling consisted of a simple feedback loop, with aerodynamic outputs providing structural inputs and structural outputs providing aerodynamic inputs. The structural model was a typical section with two degrees of freedom, pitch about the elastic axis, and plunge. The computational parameters were selected to match those of Ref. 5 and are summarized in Table 1.

For these calculations, the entire coupled system is modeled with only 10 states, 6 structural and 4 aerodynamic. Eigenvalues of the coupled, unforced system provide the frequencies of vibration. Flutter occurs when the imaginary part of the first aeroelastic frequency becomes positive. The real and imaginary parts of the first two aeroelastic frequencies, as calculated from the full and reduced numerical models, are plotted in Fig. 3. For comparison, the same calculations were performed using the full aerodynamic model. Slight deviations can be seen between the full and reduced model computations of the real part of the second modal frequency beyond the flutter boundary, but otherwise, the two methods yield equivalent results.

Computationally using the full aeroelastic model for this analysis required an eigenvalue decomposition at each reduced velocity. Each of these eigenvalue calculations was of the same complexity as solving for the balancing transformation. However, with the transformation the analysis could be done by finding eigenvalues of the reduced-order model, and this had a negligible cost compared to the initial balancing procedure.

Physical Interpretation of Balanced Aerodynamic Model

The model formulation presented here was chosen for two reasons: First, the typical section model has proven to be an excellent tool to examine basic aeroelastic behavior, and second, the model provides direct insight into some properties that appear to be inherent to a balanced realization of the relevant aerodynamics. The present section will examine the relationship between the balanced aerodynamic states and the fundamental physics of the flow. An argument is made here that the balanced coordinates can be related to indicial response pressure distributions associated with the geometry

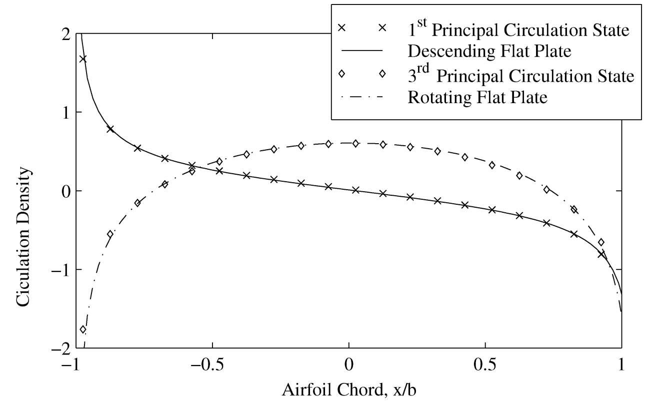

Fig. 4 Circulation distributions resulting from balanced realization compared with exact potential flow solutions.

under consideration, providing insight into more efficient numerical schemes to represent an unsteady flow about that geometry.

Model reduction is achievable primarily because the initial choice of unknowns used to model the problem is so poor. Point vortices are allowed to assume arbitrary strength from one to the next, without regard to neighboring values, known flow behavior near leading and trailing edges, or any other a priori understanding of basic aerodynamics. The balanced realization captures this inherent flow physics through its choice of primary states.

The aerodynamic states of the reduced-order model readily lend themselves to physical interpretation. Although there are four possible geometric inputs to the airfoil, α, h, $\dot{\alpha}$, and $\dot{h}$, there are only two possible input downwash distributions: uniform downwash along the entire chord due to either constant pitch angle α or plunge velocity $\dot{h}$ and linearly varying downwash due to pitch velocity $\dot{\alpha}$. Note that a constant plunge offset h does not affect the aerodynamics. When the influence of an aerodynamic wake is neglected (or a freestream velocity of zero is assumed), exact solutions exist in linearized potential flow theory for the surface vorticity distribution due to both of these downwash distributions. The first, due to a flat plate descending vertically with constant velocity, U_d, is given by

$$\gamma(x) = -2U_d \frac{(x/b)}{\sqrt{1-(x/b)^2}} \tag{7}$$

This distribution is plotted in Fig. 4. For comparison, the circulation distribution of the first (primary) balanced state, scaled appropriately, is also plotted in Fig. 4. The agreement suggests that system balancing has found the ideal combination of vortices to model the allowed airfoil motion.

For the second case of linearly varying downwash, another exact solution exists for a flat plate rotating about its midchord with constant angular velocity $\dot{\alpha}_0$:

$$\gamma(x) = \dot{\alpha}_0 b \frac{1-2(x/b)^2}{\sqrt{1-(x/b)^2}} \tag{8}$$

This distribution and the third balanced state are plotted in Fig. 4. Although there are slight deviations in this case, the match clearly demonstrates again that balancing results in a near-optimal choice of states to capture the problem physics. The remaining two balanced states are necessary for modeling the flow for long times. That is, they are significant in the steady-state solutions.

Conclusions

A method of model reduction commonly used in the field of controls was applied to a representative unsteady aerodynamic model. The method, known as balanced realization, finds a coordinate transformation to the unique set of states that have diagonal and equal controllability and observability gramian matrices. Physically, these states represent transmission paths through the aerodynamics from the system inputs, typically geometric conditions, to the system outputs, usually integrated aerodynamic forces.

This method was shown to provide substantial model reduction (two orders of magnitude) without significant loss of accuracy for representative aerodynamic and aeroelastic problems. Unlike the eigensystem approach to reduced-order modeling, which discards states based on modal damping, balanced model reduction discards states that provide little contribution to the system output. This criterion yields better performance for both transient and long time response.

A physical interpretation of the first few balanced states suggests that the optimal basis vectors are composed of solutions to simple, exact problems. Two examples shown here related transient response to a flat plate either descending or rotating with no mean flow. Such limiting-case behavior may serve as a guide in finding balanced states of more complex configurations.

The algorithm employed to calculate the appropriate balancing transformation carries the same computational cost as an eigenvalue decomposition, which is proportional to system size cubed. Thus, as problem size grows, direct calculation of the balancing transformation becomes infeasible. Special algorithms, similar to the Lanczos algorithm for finding the dominant eigenvectors and eigenvalues of very large systems, will be necessary to find the dominant balanced coordinates at reasonable cost. It is believed that the physical insight provided by the simple cases discussed here will provide a solid starting point for such algorithm development.

Acknowledgments

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A. Plotkin
Associate Editor

Effect of the Vortex Whistle on the Discharge Coefficient of Orifices

A. Jocksch*
ETH Zürich, CH-8092 Zurich, Switzerland
and
C. P. Gravett†
Rolls–Royce, plc.,
Derby, England DE24 8BJ, United Kingdom

Nomenclature

A	=	cross-sectional area
C_D	=	discharge coefficient
d	=	orifice diameter
m	=	actual mass flow rate
m_{th}	=	theoretical mass flow rate
P	=	total pressure
p	=	static pressure
Q	=	mass flow function
R	=	gas constant
Re	=	Reynolds number
T	=	temperature
t	=	orifice thickness
γ	=	ratio of specific heats

Introduction

IN the literature, discharge coefficients scatter widely for square-edged orifices, where the orifice thickness is similar to the orifice diameter. The physics of this uncertainty has been subject of discussion. Lichtarowicz et al.[1] investigated incompressible flow through such orifices. In the case of a thickness-to-diameter ratio of $t/d = 1/2$, two different flow regimes were identified. A hysteresis in the discharge coefficient was presumed to exist. Other experiments showed a train of vortex rings at certain low Reynolds numbers.[2] Hay and Spencer[3] also measured different discharge coefficients for the same arrangement. By the referencing of their measurements with compressible flow, Deckker and Chang[4] identified a hysteresis in discharge coefficients with respect to the pressure ratio at $t/d = 1/2$. When steady flow was assumed, the hysteresis effect was explained as to be the result of reattachment and nonreattachment of the flow to the orifice.[2,4]

When it is considered that flow through an orifice can generate sound,[5] it is shown that the discharge coefficient is influenced by an acoustical phenomenon, the socalled vortex whistle, a self-induced unsteadiness driven by feedback oscillations. Numerical simulations have been produced that connect this phenomenon to the discharge coefficient hysteresis mentioned earlier.

Discharge Through Orifices

The discharge coefficient is defined as the ratio of the actual mass flow rate m to the theoretical mass flow rate for isentropic flow m_{th}:

$$C_D = m/m_{\text{th}} \tag{1}$$

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*Ph.D. Student, Institute of Fluid Dynamics, Sonneggstrasse 3; jocksch@ifd.mavt.ethz.ch.

†Fluid Systems Team Leader, Trent Hall 1, P.O. Box 31.